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LETTER TO THE EDITOR

Global solution to the scalar inverse scattering problem

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Abstract. Given the scalar fields incident upon and scattered from a finite inhomogeneous region exhibiting arbitrary spatial variations of refractive index, it is shown how to construct a determinant which depends only on the refractive index within the scattering region and the fields outside this region. The analysis is simpler, and of wider applicability, than the Gel'fand–Levitan technique.

Approximate solutions to inverse scattering problems can be based on geometrical optics, on Kirchhoff's approach to diffraction theory and on the Born approximation; certain problems involving totally reflecting scatterers can be treated (apparently) exactly by analytic continuation techniques; but all reported rigorous solutions for penetrable scattering bodies derive from the method of Gel'fand and Levitan (called GL hereafter, refer to Colin 1972 for comprehensive reviews). The GL technique has not been extended beyond wave mechanical scattering from a non-local potential (Kay and Moses 1961), which is not as general as the scattering of scalar waves from a medium exhibiting arbitrary spatial variations of refractive index, as Prosser remarks in chapter 6 of Colin (1972). The intricacy of GL is quite extraordinary (cf Newton 1966), which may explain why GL has never been extended to the general case.

A comparatively simple global method is introduced here. It has the advantages that it can be applied to any kind of linear scalar wave motion and to scattering bodies that exhibit arbitrary spatial variations and are explicitly of finite size, and it can make use of scattering data spanning restricted frequency ranges (energy ranges in quantum mechanics). In order to clarify this first account as much as possible, while treating a problem beyond the capabilities of previously reported techniques, the analysis is restricted to two-dimensional macroscopic wave motion (eg the diffraction of acoustic or electrically or magnetically polarized electromagnetic cylindrical waves).

A point O in the two-dimensional space is taken as origin for the polar coordinates r and θ. A scalar field, represented by the wavefunction $\psi = \psi(r, \theta, k)$, propagates throughout space, which is free (ie, its refractive index is unity) everywhere except within a scattering region enclosed by the circle centred on O of radius a. The wavefunction satisfies

$$\nabla^2\psi + k^2 v\psi = 0. \tag{1}$$

The time factor $\exp(i\omega t)$ is suppressed, where ω is the angular frequency, $k = \omega/c$ is the wavenumber and c is the wave speed in free space. The quantity v describes the spatial inhomogeneity of the scattering region:

$$v = \begin{cases} v(r, \theta) & r < a \\ 1 & r \geqslant a. \end{cases} \tag{2}$$

It is convenient to decompose ψ into partial waves:

$$\psi = \sum_{m=-\infty}^{\infty} \psi_m(r, k) \exp(im\theta). \tag{3}$$

A known field is incident upon the scattering region and the scattered field is observed for all θ at some $r \geqslant a$. Since (1) and (2) ensure that the scattered part of ψ_m must be proportional to the 'outgoing' Hankel function of order m and argument kr, for $r \geqslant a$, it follows that the *impedance vector* $\boldsymbol{Z} = \boldsymbol{Z}(k)$ can be computed directly from observation, where the mth component of $\boldsymbol{Z}$ is

$$Z_m = Z_m(k) = \frac{\psi_m(a, k)}{\partial\psi_m(a, k)/\partial r}. \tag{4}$$

The inverse scattering problem is posed as: find v for all $r < a$, given $\boldsymbol{Z}$.

Provided that v is not zero or infinite at O—the position of O can be adjusted (thereby increasing a) to ensure this—the behaviour of ψ_m must mimic that of the cylindrical Bessel function $J_m(\gamma r)$ as $r \to 0$, where γ depends only upon k—Newton's (1966, chapter 12) discussion of three-dimensional scatterers and spherical Bessel functions can be adapted straightforwardly to the present two-dimensional case. So each ψ_m can be written as

$$\psi_m = \sum_{l=1}^{\infty} b_{l,m}(k)\phi_{l,m}(r, k), \qquad r \leqslant a \tag{5}$$

where, on account of (4) and the required behaviour of ψ_m as $r \to 0$,

$$\phi_{l,m} = J_m(h_{l,m}r/a); \qquad J_m(h_{l,m}) = (h_{l,m}/a)Z_m J'_m(h_{l,m}) \tag{6}$$

where the prime denotes the derivative. Note that $h_{l,m} = h_{l,m}(k)$ since Z_m is a function of k. The theory of Dini series (Watson 1958) indicates that, for fixed m, the $\phi_{l,m}$ form an orthogonal set in the range $0 \leqslant r \leqslant a$; and because of (6) they are *those eigenfunctions of the mth partial wave which incorporate the scattering data.*

It is convenient to write $(v-1)$ as an angular trigonometrical Fourier series

$$v = 1 + \sum_{n=-\infty}^{\infty} v_n(r) \exp(in\theta), \qquad r < a \tag{7}$$

and to express each v_n as a conventional Fourier–Bessel series (Watson 1958)

$$v_n = \sum_{p=1}^{\infty} A_{n,p} J_n(j_{n,p}r/a), \qquad J_n(j_{n,p}) = 0 \tag{8}$$

so that the inverse scattering problem reduces to finding the constants $A_{n,p}$. The quantities

$$\chi_{l,l',m,m',n,p}(k) = \int_0^a J_n(j_{n,p}r/a)\phi_{l,m}(r, k)\phi^*_{l',m'}(r, k)\, r\, dr \tag{9}$$

$$N_{l,m}(k) = \int_0^a |\phi_{l,m}(r, k)|^2 r\, dr \tag{10}$$

are needed later (the asterisk denotes the complex conjugate).

It follows from (1), (3), (5), (6), the orthogonality of the $\exp(im\theta)$ in the range $0 \leqslant \theta < 2\pi$ and the form of Bessel's differential equation that

$$-\nabla^2\psi = \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} (h_{l,m}/a)^2 b_{l,m}\phi_{l,m} \exp(im\theta), \qquad r \leqslant a \tag{11}$$

so that (7) through (10) and the orthogonality of the $\phi_{l,m}$ show that

$$\sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} \left([k^2-(h_{l',m'}/a)^2]N_{l',m'}\delta_{l,l'}\delta_{m,m'} + k^2 \sum_{p=1}^{\infty} A_{m'-m,p}\chi_{l,l',m,m',m'-m,p}\right) b_{l,m} = 0 \tag{12}$$

where l' is any non-negative integer, m' is any integer and δ_{mn} is the Kronecker delta. Now (12) is an infinite system of linear algebraic equations for the $b_{l,m}$, and it possesses a non-trivial solution only if the determinant—called here the *inverse scattering determinant*—of the coefficients of the $b_{l,m}$ is zero.

The remarkable thing about the inverse scattering determinant is that it depends only upon the unknown constants $A_{n,p}$ and the observed scattering data. The behaviour of the wavefunction within the scattering region (characterized by the $b_{l,m}$) has been eliminated. The evaluation of the $A_{n,p}$ is a multiple eigenvalue problem of non-standard type. However, unless the behaviour of v for $r < a$ is physically unreasonable, only a finite number, M say, of the $A_{n,p}$ will be needed to reconstruct v to within a given tolerance. This implies that the M significant $A_{n,p}$ can be found to the required accuracy by truncating the determinant to order M (cf Kantorovich and Krylov 1958) and examining it at M values of k, within the range of frequencies for which scattering data are available. Several interesting numerical questions have emerged and we are currently investigating them.

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